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A Property of Graphs of Polytopes

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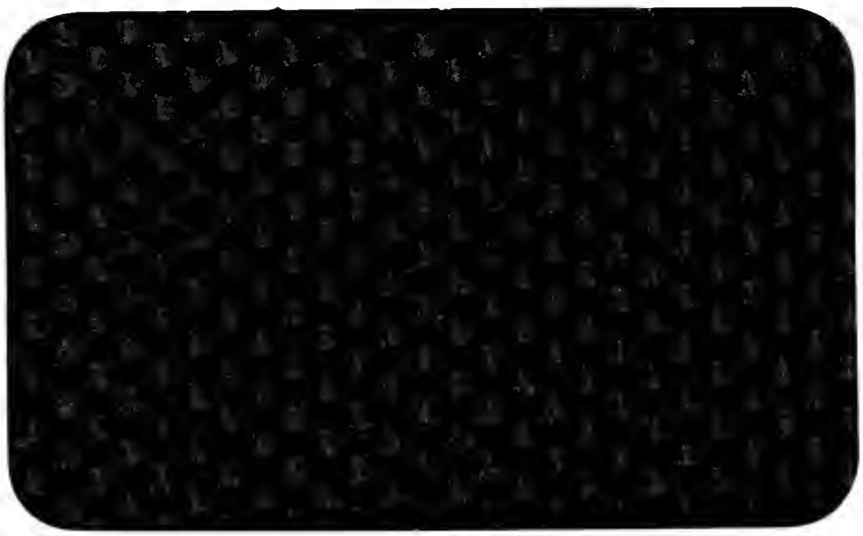

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Abstract

We prove that the subgraph obtained by removing the vertices of a k-face from the graph of a d-polytope $(0 \leq k < d)$ is $(d-k-1)$-connected. Further we show that this lower bound is tight for $k < d-1$. We also show that for $k = d-1$ the known lower bound is tight.

1 Introduction

The 1-skeleton of a polytope P is called the *graph of P* and is denoted $G(P)$. A celebrated result of Balinski shows that *the graph of every d-polytope is d-connected* [1]. The central idea in Balinski's proof is that, *if at a vertex v, a linear functional does not attain the maximum of all its values in the polytope, then v must be adjacent to a vertex at which the linear functional has a higher value.* Using this fact one can easily show that *removing the vertices of a proper face does not disconnect the graph of the polytope* [2]. This corollary provides the background for our discussion.

We address the problem of determining the best lower bound on the connectivity of the remaining subgraph when the vertices of a proper face are removed from the graph of a polytope. In this paper we settle the problem completely. In section 2 we prove the following theorem.

> **Theorem 1 :** *Let P be a d-polytope and Z a k-face of P, $0 \leq k \leq d-1$. Let $G(P)$ and $G(Z)$ be the graphs of P and Z respectively. Then the complement of $G(Z)$ (i.e. the subgraph of $G(P)$ induced by the vertices that are not in Z) is $(d-k-1)$-connected.*

Furthermore, we show that this is a tight lower bound for d-polytopes, when $k < d-1$. That is, for every $d \in \mathcal{N}$ we can construct a d-polytope which has for every $0 \leq k < d-1$, a k-face Z such that the complement of $G(Z)$ is *not* $(d-k)$-connected. When $k = d-1$ however the aforementioned corollary of Balinski's proof suggests a better lower bound which we prove is tight. That is, for every $d \in \mathcal{N}$ we can construct a d-polytope which contains a facet F such that the complement of $G(F)$ is *not* 2-connected.

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2 Connectivity of a subgraph induced by a coface

Let Γ be an abstract graph. Then $V(\Gamma)$ denotes the vertex set of Γ and $E(\Gamma)$ denotes its edge set. Let $V' \subseteq V(\Gamma)$. Let Γ' be the subgraph of Γ induced by V'. By the *complement* of Γ', denoted $\Gamma \setminus \Gamma'$, we mean the subgraph of Γ induced by the vertex set $V(\Gamma) \setminus V(\Gamma')$.

Let P be a d-polytope and F a proper face of P. Then $V(P)$ and $V(F)$ are the vertex sets of P and F respectively. $G(P)$ is the graph (1-skeleton) of P and $G(F)$ the subgraph of $G(P)$ induced by the vertex set $V(F)$. A subset $C \subset V(P)$ is called *a coface of* P if $F = conv\,(V(P) \setminus C)$ is a face of P.

We use the following three results to prove theorem 1.

Result 2.1 ([1,2]) *If M is a proper face of a polytope Q, then the complement of $G(M)$ (i.e. $G(Q) \setminus G(M)$) is connected.*

Result 2.2 ([3]) *A graph $G = (V, E)$ with $|V| \geq k + 1$ is k-connected, if and only if it satisfies the following equivalent conditions.*

1. *If we remove any $k - 1$ vertices from V the subgraph induced by the remaining vertices is still connected.*
2. *Between any two vertices in G there are at least k vertex-disjoint paths.*

Result 2.3 ([3]) *Let $G = (V, E)$ be a graph and let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two subgraphs of G that are k-connected. In addition let V_1 and V_2 have at least k vertices in common. Then the union of G_1 and G_2*

$$G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$$

is also k-connected.

Theorem 1 *Let P be a d-polytope and Z a k-face of P, $0 \leq k \leq d - 1$. Then the complement of $G(Z)$ (i.e. $G(P) \setminus G(Z)$) is $(d - k - 1)$-connected.*

Proof : The proof of the theorem is by induction on the dimension d. The theorem is trivial when $d = 1$ or $d = 2$. Consider a $d > 2$ and assume that the theorem is true for all n-polytopes, when $1 \leq n < d$. Let P be a d-polytope and Z a k-face of P, $0 \leq k \leq d - 1$. If $k = d - 1$ there is nothing to prove. If $k = d - 2$ we complete the proof at once by appealing to result 2.1. So we assume that $0 \leq k < d - 2$.

Let

$$\mathcal{F} = \{X \mid X \text{ is a facet of } P\}.$$

We partition the set $\mathcal{F}$ into 2 classes $\mathcal{A}$ and $\mathcal{B}$ as follows:

$$\mathcal{A} = \{Y \mid Y \in \mathcal{F};\ Y \text{ contains } Z\}$$

$$\mathcal{B} = \{W \mid W \in \mathcal{F};\ W \text{ does not contain } Z\}.$$

Note that a facet in $\mathcal{B}$ could have a non-empty intersection with Z. Also observe that both $\mathcal{A}$ and $\mathcal{B}$ are non-empty.

Let $\mathcal{C} \subseteq \mathcal{F}$. Then

$$V(\mathcal{C}) = \bigcup_{F \in \mathcal{C}} V(F).$$

$G(\mathcal{C})$ and $G_Z(\mathcal{C})$ denote the subgraphs of $G(P)$ induced by the vertex sets $V(\mathcal{C})$ and $V(\mathcal{C}) \setminus V(Z)$ respectively.

Consider a facet $X \in \mathcal{B}$. Let $T = X \cap Z$. T is a face of X and $dim(T) \leq k-1$. Therefore [4], X contains a $(d-k-1)$-face that does not intersect T (and hence disjoint from Z). This shows that $G_Z(X)$ (and hence $G_Z(\mathcal{B})$ and $G_Z(\mathcal{F})$) has at least $d-k$ vertices; so it is meaningful to consider $(d-k-1)$-connectedness of $G_Z(\mathcal{B})$ and $G_Z(\mathcal{F})$ (lemmas 2.3 and 2.8).

We first show that $G_Z(\mathcal{B})$ is $(d-k-1)$-connected. To do so we need the following notion of 'connectivity' of facets. A subset $\mathcal{C} \subseteq \mathcal{F}$ is said to form a *'connected' complex* if for every $X, Y \in \mathcal{C}$ there exists a sequence of facets

$$X = S_1, \cdots\cdots, S_m = Y$$

such that

i) $S_j \in \mathcal{C}$, for $1 \leq j \leq m$ and
ii) for $1 \leq j \leq m-1$, $S_j \cap S_{j+1}$ is a $(d-2)$-face of P.

To distinguish this notion of 'connectivity' from *edge connectivity*, in addition to appealing to the context we use quotes in the former case.

Lemma 2.1 *$\mathcal{B}$ is a 'connected' complex.*

Proof : Consider a dual P^* of P. Let the k-face Z of P correspond to the $(d-k-1)$-face Z^* in P^*. Let X be some facet in $\mathcal{B}$ and X^* the corresponding vertex in P^*. Since X does not contain Z, X^* is not contained in Z^*. Thus the class $\mathcal{B}$ in P corresponds to the set $\mathcal{B}^*$ of all the vertices in P^* that are not contained in Z^*. Since Z^* is a proper face of P^*, using result 2.1 we conclude that the subgraph of $G(P^*)$ induced by $\mathcal{B}^*$ is connected. Therefore $\mathcal{B}$ is a 'connected' complex. ○

Lemma 2.2 *For every facet $X \in \mathcal{B}$, $G_Z(X)$ is $(d-k-1)$-connected.*

Proof : Let $X \in \mathcal{B}$. Let

$$M = X \cap Z.$$

Since X does not contain Z and since the intersection of any two faces of P is again a face of P, M is either a proper face of Z or M is an empty face. In either case $dim(M) \leq k-1$. Since X is a $(d-1)$-polytope, from the inductive hypothesis we know that the subgraph of $G(X)$ induced by $V(X) \setminus V(M)$ (i.e. $G_Z(X)$) is at least $(d-k-1)$-connected. ○

Lemma 2.3 *$G_Z(\mathcal{B})$ is $(d-k-1)$-connected.*

Proof : Let $|\mathcal{B}| = n$ be the number of facets in $\mathcal{B}$. Given a subset $\mathcal{S} \subset \mathcal{B}$ which has fewer than n facets, such that $G_Z(\mathcal{S})$ is $(d-k-1)$-connected, we show that we can add one more facet $X \in \mathcal{B} \setminus \mathcal{S}$, to $\mathcal{S}$ such that $G_Z(\mathcal{S} \cup \{X\})$ is $(d-k-1)$-connected. So if we start with $\mathcal{S}$ containing a single facet from $\mathcal{B}$, knowing from the preceding lemma that $G_Z(\mathcal{S})$ is $(d-k-1)$-connected to start with, we can add facets to $\mathcal{S}$ one at a time in the aforementioned way until $\mathcal{S} = \mathcal{B}$ thus showing $G_Z(\mathcal{B})$ is $(d-k-1)$-connected.

Consider an $\mathcal{S} \subset \mathcal{B}$, such that $G_Z(\mathcal{S})$ is $(d-k-1)$-connected and $1 \leq |\mathcal{S}| < n$. Since $\mathcal{B}$ is a 'connected' complex, there is an $X \in \mathcal{B} \setminus \mathcal{S}$ that shares a $(d-2)$-face with some $Y \in \mathcal{S}$. Let

$$W = X \cap Y$$

$$R = X \cap Y \cap Z.$$

R and W are faces of P. Since neither X nor Y contains the k-face Z, $dim(R) \leq k-1$. Therefore [4], W contains a face T such that

$$T \cap R = \emptyset$$

and

$$dim(T) = d - 2 - dim(R) - 1 \geq d - k - 2.$$

So, $G_Z(Y)$ and $G_Z(X)$ (and hence $G_Z(\mathcal{S})$ and $G_Z(X)$) have at least $d-k-1$ vertices in common. Moreover since both the subgraphs $G_Z(\mathcal{S})$ and $G_Z(X)$ of $G(P)$ are $(d-k-1)$-connected, it follows from result 2.3 that their union is $(d-k-1)$-connected. But

$$V(\,G_Z(\mathcal{S}) \,\cup\, G_Z(X)\,) = V(\,G_Z(\,\mathcal{S} \,\cup\, \{X\}\,)\,)$$

and

$$E(\,G_Z(\mathcal{S}) \,\cup\, G_Z(X)\,) \subseteq E(\,G_Z(\,\mathcal{S} \,\cup\, \{X\})\,).$$

Therefore it follows that $G_Z(\mathcal{S} \,\cup\, \{X\})$ is $(d-k-1)$-connected; that completes the proof. ○

Before turning to set $\mathcal{A}$, it is convenient to prove the following lemma.

Lemma 2.4 *Let v be a vertex of a d-polytope Q and let F be some proper face of Q that contains v. Then v has at least one neighbouring vertex in P that does not belong to F.*

Proof : A vertex of a d-polytope together with all its neighbours affinely spans the entire space, E^d [3]. Hence a proper face cannot contain all the neighbours of a vertex in it. ○

Lemma 2.5 *Every facet in $\mathcal{A}$ is 'adjacent' to at least one facet in $\mathcal{B}$ along a $(d-2)$-face.*

Proof : The k-face Z of P corresponds to the $(d-k-1)$-face Z^* of P^*; the set $\mathcal{A}$ corresponds to the vertex set $V(Z^*)$ of Z^* and the set $\mathcal{B}$ corresponds to the set of all those vertices of P^* that are not in $V(Z^*)$. Since Z^* is a proper face of P^*, from the previous lemma it follows that every vertex in Z^* is adjacent to at least one vertex that is not in Z^*. ○

Lemma 2.6 *Let $X \in \mathcal{A}$. Then $G_Z(X)$ and $G_Z(\mathcal{B})$ have at least $d-k-1$ vertices in common.*

Proof : From the preceding lemma we know that there is a facet Y in $\mathcal{B}$ such that X and Y share a $(d-2)$-face T, in P. T (a face of Y) shares a face of dimension at most $k-1$, with Z. Therefore T contains a $(d-k-2)$-face that does not intersect Z. This means T has at least $d-k-1$ vertices none of which is a vertex of Z. So $G_Z(X)$ and $G_Z(Y)$ (and hence $G_Z(X)$ and $G_Z(\mathcal{B})$) share at least $d-k-1$ vertices. ○

Lemma 2.7 *Let X be a facet in $\mathcal{A}$. Let $v, w \in V(X) \setminus V(Z)$ be any two vertices of X. Then there are at least $d-k-1$ vertex-disjoint paths between v and w, in $G_Z(\mathcal{F})$.*

Proof : X is a $(d-1)$-polytope and Z is a k-face of X. From the inductive hypothesis, we conclude that $G_Z(X)$ is $(d-k-2)$-connected. Therefore there are $d-k-2$ vertex-disjoint paths between v and w, in $G_Z(X)$. Let $n(v), n(w) \in V(P) \setminus V(X)$ be neighbours of v and w respectively (these exist by lemma 2.4). From result 2.1 we know that there exists an edge path Π between $n(v)$ and $n(w)$ that does not pass through any vertex in $V(X)$. Since Z is contained in X and since Π misses X, Π is a path in $G_Z(\mathcal{F})$. The path

$$v \leftrightarrow n(v) \overset{\Pi}{\longrightarrow \cdots\cdots \longrightarrow} n(w) \leftrightarrow w$$

is vertex-disjoint from every path between v and w in $G_Z(X)$. Together with the $d-k-2$ paths in $G_Z(X)$ we have $d-k-1$ vertex-disjoint paths in all, between v and w. ○

The following lemma completes the proof of the theorem.

Lemma 2.8 *$G_Z(\mathcal{F})$ is $(d-k-1)$-connected.*

Proof : Suppose $G_Z(\mathcal{F})$ is not $(d-k-1)$-connected. Then we can remove a set of $d-k-2$ vertices, say $v_1, \ldots, v_{d-k-2}$, from $G_Z(\mathcal{F})$ such that the remaining subgraph is not connected. Let

$$W = \{v_1, \ldots, v_{d-k-2}\}.$$

If $\mathcal{C} \subseteq \mathcal{F}$, then $G_{Z,W}(\mathcal{C})$ denotes the subgraph of $G(P)$ induced by the vertex set $(V(\mathcal{C}) \setminus V(Z)) \setminus W$.

Let $C_1, \cdots\cdots, C_r$ $(r \geq 2)$ be the connected components of $G_{Z,W}(\mathcal{F})$. Since $G_Z(\mathcal{B})$ is $(d-k-1)$-connected (lemma 2.3) $G_{Z,W}(\mathcal{B})$ is connected; so $G_{Z,W}(\mathcal{B})$ is contained in some connected component, say C_1. Consider a component C_2 that is different from C_1. Let v be a vertex in C_2. Since $v \notin V(C_1)$, $v \notin V(\mathcal{B})$. So $v \in V(\mathcal{A}) \setminus V(\mathcal{B})$. Since $v \in V(\mathcal{A})$, v is a vertex of a facet $Y \in \mathcal{A}$. Lemma 2.6 asserts that $G_Z(Y)$ and $G_Z(\mathcal{B})$ have at least $d-k-1$ vertices in common. Let

$$U = \{m_1, \ldots\ldots, m_q\} \qquad q \geq d-k-1$$

be the set of vertices shared by $G_Z(Y)$ and $G_Z(\mathcal{B})$. Since we removed only $d-k-2$ vertices (when we removed the set W from $G_Z(\mathcal{F})$) at least one vertex of U, say z, remains in $G_{Z,W}(\mathcal{F})$. Both v and z are contained in the facet $Y \in \mathcal{A}$. By appealing to lemma 2.7 we conclude that there are at least $d-k-1$ vertex-disjoint paths between v and z, in $G_Z(\mathcal{F})$. Removing the $d-k-2$ vertices of W from $G_Z(\mathcal{F})$ will leave at least one of these paths between x and z, intact. Therefore x and z must lie in the same connected component in $G_{Z,W}(\mathcal{F})$. But $z \in V(\mathcal{B})$ and hence z must lie in C_1 and we assumed that v lies in C_2. Contradiction. So we conclude that $G_{Z,W}(\mathcal{F})$ cannot have

more than one connected component. That is, removing any $d-k-2$ vertices does not disconnect $G_Z(\mathcal{F})$. Hence $G_Z(\mathcal{F})$ is $(d-k-1)$-connected. ○ ○

Theorem 1 shows that if we remove the vertices of a k-face from the graph of a d-polytope the remaining subgraph is *at least* $(d-k-1)$-connected. The following construction shows that this lower bound is tight when $k < d-1$. (We do not use a d-simplex instead of the following construction, because removing the vertices of a k-face leaves only $d-k$ vertices in the graph of the simplex and it would not be meaningful to consider the $(d-k)$-connectedness of the remaining subgraph).

Construction 1 : *Let P be a simple d-polytope and v a vertex of P. Obtain Q by truncating the vertex v from P. That is, if H is a hyperplane such that $v \in H^+$ and $V(P) \setminus \{v\} \subset H^-$ (H^+ and H^- are the two open half-spaces determined by H) then*

$$Q = (H \cup H^-) \cap P.$$

Q is a simple d-polytope. $F = H \cap Q$ is a facet of Q. More importantly, F is a $(d-1)$-simplex. Consider any k-face Z of F, $0 \leq k < d-1$. Since Z is a k-simplex it has $k+1$ vertices. Since $k < d-1$ there is a vertex $z \in V(F) \setminus V(Z)$. z has d neighbours in Q and the vertices of Z are $k+1$ of them. So in the subgraph induced by the vertex set $V(Q) \setminus V(Z)$, z has $d-k-1$ neighbours and hence the subgraph $G(Q) \setminus G(Z)$ cannot be $(d-k)$-connected. ○

From result 2.1 we know that the set of all vertices that are not in a given facet induce a subgraph that is *at least* 1-connected. We conclude by showing that this lower bound is tight as well.

Construction 2 : *Let X be any $(d-1)$-polytope contained in a hyperplane H in R^d. Choose $v_1, v_2 \in R^d \cap H^+$ such that v_1 and v_2 are vertices of $Y = conv(\{v_1, v_2\} \cup X)$. Now choose a $v_3 \in H^+ \setminus Y$ such that $conv(v_1, v_3) \cap int(Y) \neq \emptyset$.*

It is easy to see that v_1, v_2 and v_3 are vertices of $Q = conv(\{v_3\} \cup Y)$. Moreover we chose v_3 such that (v_1, v_3) is not an edge of Q. X is a facet of Q and the complement of $G(X)$ (i.e. $G(Q) \setminus G(X)$) is the path $v_1 \leftrightarrow v_2 \leftrightarrow v_3$ which is *not* 2-connected. ○

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